Algebra III
Semestral Exam
November 29, 2002

Instructor: S. P. Inamdar

Instructions. All fields are assumed to be of characteristic zero and all field extensions are finite.

1. (a) Let α be algebraic over a field F. Prove that $[F(\alpha) : F]$, the dimension of $F(\alpha)$ as a F-vector space, equals the degree of the irreducible polynomial of α over F. (6)
   (b) Find the irreducible polynomial of $i + 4\sqrt{2}$ over Q. (4)
2. (a) Let $K \subset L \subset M$ be fields. Prove that $[M : L][L : K] = [M : K]$. (6)
   (b) Let F, α be as in Q.1(a). If $[F(\alpha) : F]$ is odd, prove that $F(\alpha^2) = F(\alpha)$. (4)
3. (a) Let f be an irreducible cubic equation over F, and let δ be the square root of the discriminant of f. Prove that f remains irreducible over the field $F(\delta)$. (5)
   (b) Let K be the splitting field of a polynomial f with distinct roots $\alpha_1, \dots, \alpha_n$. Then the Galois group G of f may be regarded as a subgroup of the symmetric group S_n. Prove that a change of numbering of the roots changes G to a conjugate subgroup. (5)
4. Let K be a Galois extension of F with Galois group G.
   (a) For any subgroup H of G, prove that there exists $\beta \in K$ whose stabiliser is H. (5)
   (b) Let f be an irreducible polynomial over F and g and h be two of its irreducible factors over K. Prove that the degree of g equals that of h. (5)
5. (a) Compute the discriminant of $x^4 + 1$ and compute its Galois group over Q. (5)
   (b) Let f be an irreducible quartic polynomial over Q with exactly two real roots. What can you say about its Galois group over Q? (5)
6. (a) Let $a \in F$ and p be a prime. Suppose that the polynomial $x^p - a$ is reducible in $F[x]$. Prove that it has a root in F. (5)
   (b) Let ξ_{11} denote the primitive 11th root of unity. Find an element that generates a subfield of $Q(\xi_{11})$ having degree 5 over Q and find its equation. (5)
7. (a) Let ξ_n be the primitive nth root of unity for an integer n. Prove that $Q(\xi_n)$ is a Galois extension of Q. (5)
   (b) Sketch the outline of the proof that every quadratic extension of Q is contained in a cyclotomic extension of Q. (5).